Solving Systems of Linear Equations

Objective

Learn three ways to solve systems of linear equations.

- Gaussian elimination
- Inverse matrix
- Cramer's rule

1 Gaussian Elimination

Objective

- Learn how to replace a system of linear equations with an **augmented matrix**.
- Learn how to use different elimination methods (**scaling**, **replacement**, and **swap**)
- Know when to stop (obtain the **Reduced Row Echelon Form)**

1.1 Overall Goal

Our overall goal here is to use Gaussian Elimination to change the original equation system to

$$\begin{cases} x_1 = a & (1) \\ x_2 = b & (2) \\ x_3 = c & (3) \end{cases} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

which means that for (1) we need to eliminate x_1 in (2) and (3), x_2 in (1) and (3), and x_3 in (1) and (2).

1.2 Approaches

Step 1. Treat a system of linear equations as matrix form, $AX = d$. Then translate them into an augmented matrix.

As equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = d_1 \quad (1) \\ & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n & = d_n \quad (n) \end{cases}$$

$\Longrightarrow$

A matrix storing just the coefficients:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & d_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & d_n \end{array}\right]$$

Step 2. Apply elimination methods (row operations) to find the reduced row echelon form, and get the solution.

- Scaling. Multiply all elements in a row by a nonzero number.
- Replacement. Add a multiple of one row to another, replacing the second row with the result.
- Swap. Interchange the position of rows.

Finally, we can find the solutions as a **Reduced Row Echelon** form.

$$\left[\begin{array}{cccc|c} 1 & 0 & \ldots & 0 & x_1 \\ 0 & 1 & \ldots & 0 & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & x_n \end{array}\right]$$

Note

The elementary row operations do not change the solution set of the corresponding system of linear equations.

1.3 Example 1

As equations:

$$\begin{cases} 4x_1 + x_2 - 5x_3 = 8 & (1) \\ -2x_1 + 3x_2 + x_3 = 12 & (2) \\ 3x_1 - x_2 + 4x_3 = 5 & (3) \end{cases} \implies$$

Associated matrix:

$$\left[\begin{array}{ccc|c} 4 & 1 & -5 & 8 \\ -2 & 3 & 1 & 12 \\ 3 & -1 & 4 & 5 \end{array}\right]$$

Step 1. Start with (1). Notice that we can obtain the coefficient of x_1 as 1 by subtracting (3) from (1). Keep in mind that doing this will only change the coefficients in (1).

$$(1) - (3) \rightarrow \text{new } (1)$$

As equations:

$$\begin{cases} x_1 + 2x_2 - 9x_3 = 3 & (1) \\ -2x_1 + 3x_2 + x_3 = 12 & (2) \\ 3x_1 - x_2 + 4x_3 = 5 & (3) \end{cases} \implies$$

Associated matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -9 & 3 \\ -2 & 3 & 1 & 12 \\ 3 & -1 & 4 & 5 \end{array}\right]$$

Step 2. Now we want to eliminate x_1 in (2) and (3). How? Since we have x_1 with coefficient 1 in (1), we can multiply (1) by 2 and add it to (2) to eliminate x_1 in (2).

$$(1) \times 2 + (2) \rightarrow \text{new } (2)$$

As equations:

$$\begin{cases} x_1 + 2x_2 - 9x_3 = 3 & (1) \\ \qquad 7x_2 - 17x_3 = 18 & (2) \\ 3x_1 - x_2 + 4x_3 = 5 & (3) \end{cases} \implies$$

Associated matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -9 & 3 \\ 0 & 7 & -17 & 18 \\ 3 & -1 & 4 & 5 \end{array}\right]$$

Similarly, we do row operations to row 3.

$$1 \times (-3) + 3 \rightarrow \text{new } 3$$

As equations:

$$\begin{cases} x_1 + 2x_2 - 9x_3 = 3 & 1 \\ 7x_2 - 17x_3 = 18 & 2 \\ -7x_2 + 31x_3 = -4 & 3 \end{cases}$$

$\Longrightarrow$

Associated matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -9 & 3 \\ 0 & 7 & -17 & 18 \\ 0 & -7 & 31 & -4 \end{array}\right]$$

Step 3. Now we can add 2 to 3 to directly to eliminate x_2 in 3. By doing this, we are able to obtain x_3.

$$2 + 3 \rightarrow \text{new } 3$$

As equations:

$$\begin{cases} x_1 + 2x_2 - 9x_3 = 3 & 1 \\ 7x_2 - 17x_3 = 18 & 2 \\ 14x_3 = 14 & 3 \end{cases}$$

$\Longrightarrow$

Associated matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -9 & 3 \\ 0 & 7 & -17 & 18 \\ 0 & 0 & 14 & 14 \end{array}\right]$$

Simplify for row 3, we have

$$3 \div 14 \rightarrow \text{new } 3$$

As equations:

$$\begin{cases} x_1 + 2x_2 - 9x_3 = 3 & 1 \\ 7x_2 - 17x_3 = 18 & 2 \\ x_3 = 1 & 3 \end{cases}$$

$\Longrightarrow$

Associated matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -9 & 3 \\ 0 & 7 & -17 & 18 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Step 4. Now we are able to use new 3 to eliminate x_3 in 2. This is equivalent to plugging $x_3 = 1$ to 2.

$$3 \times 17 + 2 \rightarrow \text{new } 2$$

As equations: Associated matrix:

$$\begin{cases} x_1 + 2x_2 - 9x_3 = 3 & 1 \\ 7x_2 = 35 & 2 \\ x_3 = 1 & 3 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -9 & 3 \\ 0 & 7 & 0 & 35 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Simplify the result in 2, we have

$$2 \div 7 \rightarrow \text{new } 2$$

As equations: Associated matrix:

$$\begin{cases} x_1 + 2x_2 - 9x_3 = 3 & 1 \\ x_2 = 5 & 2 \\ x_3 = 1 & 3 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -9 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Step 5. Finally, we use 2 3 to eliminate x_2 and x_3 in 1 to get x_1.

$$3 \times 9 + 1 \rightarrow \text{new } 1$$

As equations: Associated matrix:

$$\begin{cases} x_1 + 2x_2 = 12 & 1 \\ x_2 = 5 & 2 \\ x_3 = 1 & 3 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$2 \times (-2) + 1 \Rightarrow \text{new } 1$$

As equations:

$$\begin{cases} x_1 & & & = 2 \quad 1 \\ & x_2 & & = 5 \quad 2 \\ & & x_3 & = 1 \quad 3 \end{cases} \Longrightarrow$$

Associated matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, the solutions are

$$\begin{cases} x_1 = 2 \\ x_2 = 5 \\ x_3 = 1 \end{cases}$$

1.4 Example 2

As equations:

$$\begin{cases} x_1 \quad - 5x_3 = 1 & (1) \\ x_2 + x_3 = 4 & (2) \\ 2x_2 + 2x_3 = 8 & (3) \end{cases} \Longrightarrow$$

Associated matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 2 & 2 & 8 \end{array}\right]$$

Step 1. We check the coefficient of x_1 in (1), (2), and (3). They are 1,0, and 0, meaning that we do not need to do the elimination for x_1.

Step 2. We check the coefficient of x_2 in these three equations. They are 0, 1, and 2. Ideally, We want to multiply (2) with (-2) and add it to (3) to eliminate x_2 in (3). However, this will give us $0 = 0$, implying that (2) and (3) are two identical equations. We only need to keep one of them. Let's keep (2).

As equations:

$$\begin{cases} x_1 \quad - 5x_3 = 1 & (1) \\ x_2 + x_3 = 4 & (2) \end{cases} \Longrightarrow$$

Associated matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Step 3. Notice that now we have two equations but three unknowns, meaning that we will not obtain the unique solution. However, we can express x_1 and x_2 using x_3 and x_3 are called the "Free variables".

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \end{cases}$$

Since x_3 is a free variable, this system of equations has infinite solutions. x_3 can take any values, and for each value it takes, x_1 and x_2 will be assigned values accordingly.

2 Inverse Matrix

Objective

- Know why this method works

  $AX = d \Longleftrightarrow X = A^{-1}d$

- Know when to use this method

  Matrix A has to be invertible ($det(A) \neq 0$)

2.1 Example

$$\begin{cases} 4x_1 + x_2 - 5x_3 = 8 \\ -2x_1 + 3x_2 + x_3 = 12 \\ 3x_1 - x_2 + 4x_3 = 5 \end{cases}$$

Step 1. Translate this system into matrix form.

$$AX = d$$

As equations: Associated matrix form:

$$\begin{cases} 4x_1 + x_2 - 5x_3 = 8 \\ -2x_1 + 3x_2 + x_3 = 12 \\ 3x_1 - x_2 + 4x_3 = 5 \end{cases} \Longrightarrow \underbrace{\begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}}_{\text{Coefficient matrix } A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{matrix } X} = \underbrace{\begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}}_{\text{Matrix } d}$$

The solution can be calculated by

$$X = A^{-1}d = \frac{1}{|A|} adjAd$$

Step 2. Compute the inverse matrix of A. We can calculate the determinant $|A|$ using **Laplace expansion** and adjoint matrix of matrix A

$$\begin{aligned} |A| &= a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}| \\ &= 98 \neq 0 \end{aligned}$$

$$adjA = \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}$$

Step 3. Use $X = A^{-1}d$ to obtain the solutions.

$$X = \frac{1}{|A|} adjAd = \frac{1}{98} \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

3 Cramer's Rule

3.1 Definition

We already know how to translate a system of linear equations into a matrix form. To find the solution value of x_j, we can merely replace the jth column of the determinant of coefficient matrix $|A|$ by the constant terms d_1, d_2,$\cdots$,d_n to get a new determinant $|A_j|$ and then divide $|A_j|$ by the original determinant $|A|$.

The solution of the system $AX = d$ can be expressed as:

$$x_j = \frac{|A_j|}{|A|} = \frac{1}{|A|} \underbrace{\begin{vmatrix} a_{11} & a_{12} & \dots & \boxed{d_1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & d_2 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & d_n & \dots & a_{nn} \end{vmatrix}}_{|A|_j}$$

3.2 Example 1

$$\begin{cases} 5x_1 + 3x_2 = 30 \\ 6x_1 - 2x_2 = 8 \end{cases}$$

Step 1. Calculate the determinant of the coefficient matrix A.

$$A = \begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}$$

$$|A| = 5 \times (-2) - 3 \times 6 = -28 \neq 0$$

Step 2. Solve x_1 using Cramer's Rule.

$$x_1 = \frac{|A_1|}{|A|} = \frac{1}{|A|}\begin{vmatrix} \boxed{\begin{matrix} d_1 \\ d_2 \end{matrix}} & \begin{matrix} a_{12} \\ a_{22} \end{matrix} \end{vmatrix} = -\frac{1}{28}\begin{vmatrix} \boxed{\begin{matrix} 30 \\ 8 \end{matrix}} & \begin{matrix} 3 \\ -2 \end{matrix} \end{vmatrix} = -\frac{1}{28}(30 \times (-2) - 8 \times 3) = -\frac{-84}{28} = 3$$

Step 3. Solve x_2.

$$x_2 = \frac{|A_2|}{|A|} = \frac{1}{|A|}\begin{vmatrix} \begin{matrix} a_{11} \\ a_{21} \end{matrix} & \boxed{\begin{matrix} d_1 \\ d_2 \end{matrix}} \end{vmatrix} = -\frac{1}{28}\begin{vmatrix} \begin{matrix} 5 \\ 6 \end{matrix} & \boxed{\begin{matrix} 30 \\ 8 \end{matrix}} \end{vmatrix} = -\frac{1}{28}(5 \times 8 - 6 \times 30) = -\frac{-140}{28} = 5$$

Thus, the solution of this system of equations is $x_1 = 3$ and $x_2 = 5$.

3.3 Example 2

$$\begin{cases} 4x_1 + x_2 - 5x_3 = 8 \\ -2x_1 + 3x_2 + x_3 = 12 \\ 3x_1 - x_2 + 4x_3 = 5 \end{cases}$$

Step 1. Calculate the determinant of A.

$$A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}$$

$$|A| = 98 \neq 0$$

Step 2. Solve x_1 using Cramer's Rule.

$$x_1 = \frac{|A_1|}{|A|} = \frac{1}{|A|}\begin{vmatrix} d_1 & a_{12} & a_{13} \\ d_2 & a_{22} & a_{13} \\ d_3 & a_{32} & a_{33} \end{vmatrix} = \frac{1}{98}\begin{vmatrix} 8 & 1 & -5 \\ 12 & 3 & 1 \\ 5 & -1 & 4 \end{vmatrix} = \frac{196}{98} = 2$$

Step 3. Solve x_2 using Cramer's Rule.

$$x_2 = \frac{|A_2|}{|A|} = \frac{1}{|A|}\begin{vmatrix} a_{11} & d_1 & a_{13} \\ a_{12} & d_2 & a_{13} \\ a_{13} & d_3 & a_{33} \end{vmatrix} = \frac{1}{98}\begin{vmatrix} 4 & 8 & -5 \\ -2 & 12 & 1 \\ 3 & 5 & 4 \end{vmatrix} = \frac{490}{98} = 5$$

Step 4. Solve x_3 using Cramer's Rule.

$$x_3 = \frac{|A_3|}{|A|} = \frac{1}{|A|}\begin{vmatrix} a_{11} & a_{12} & d_1 \\ a_{12} & a_{22} & d_2 \\ a_{13} & a_{23} & d_3 \end{vmatrix} = \frac{1}{98}\begin{vmatrix} 4 & 1 & 8 \\ -2 & 3 & 12 \\ 3 & -1 & 5 \end{vmatrix} = \frac{98}{98} = 1$$

Thus, the solution of this system of equations is $x_1 = 2$, $x_2 = 5$, and $x_3 = 1$.

4 Additional Notes

1. Inverse Matrix and Cramer's Rule can only be applied when $|A| \neq 0$. If $|A| = 0$, the system of equations does not have a unique solution. Pivot and rank can be used to judge whether it has infinite solutions or no solutions.
2. When applying Gaussian Elimination, remember to do the operation for one equation each time. For example, if we multiply 2 by equation (1) and add it to equation (2), then only coefficients in equation (2) change. Coefficients in equation (1) should stay the same.

3. Linear independence

Definition

Vectors $\mathbf{v_1}$, $\mathbf{v_2}$,..., $\mathbf{v_k}$ are **linearly dependent** if and only if there exist scalars c_1, c_2, ..., c_k, *not all zero* , such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_k\mathbf{v_k} = \mathbf{0}$$

Vectors $\mathbf{v_1}$, $\mathbf{v_2}$,..., $\mathbf{v_k}$ are **linearly independent** if and only if $c_1\mathbf{v_1}+c_2\mathbf{v_2}+\ldots+c_k\mathbf{v_k} = \mathbf{0}$ for scalars c_1, c_2, ..., c_k implies that $c_1=c_2=\ldots=c_k = 0$.

3.1 Example. Now we look at an example and try to check whether the column vectors are independent or not. The three vectors are

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \text{and } \mathbf{v_3} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

To use the definition above, start with the equation

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and solve this system for all possible values of c_1, c_2, and c_3. If the only solution to satisfy the system of equations above is $c_1 = c_2 = c_3 = 0$, then vectors $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ are linearly independent. Otherwise (if c_1, c_2, and c_3 are not all zero), then vectors $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ are linearly dependent. Multiply the system above out yields

$$\begin{cases} 1c_1 + 4c_2 + 7c_3 = 0 \\ 2c_1 + 5c_2 + 8c_3 = 0 \\ 3c_1 + 6c_2 + 9c_3 = 0 \end{cases}$$

This is a homogeneous linear system of equations with three unknowns c_1, c_2, and c_3. It has either trivial solution ($c_1 = c_2 = c_3 = 0$) or infinite many solutions. We reduce the coefficient matrix to its row echelon form:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Because its row echelon form has a row of zeros, the coefficient matrix is singular and therefore that the system has infinite many nonzero solutions. One such solution is easily to be

$$c_1 = 1, \ c_2 = -2, \ \text{and} \ c_3 = 1.$$

We conclude that vectors $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ are linearly dependent.